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Abstract
The assumptions required for the derivation of Bell inequalities are not satisfied for random field models in which there are *any* thermal or quantum fluctuations, in contrast to the general satisfaction of the assumptions for classical two point particle models. Classical random field models that explicitly include the effects of quantum fluctuations on measurement are possible for experiments that violate Bell inequalities.

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1. Introduction

Bell [1, chapter 7, originally 1976] shows that from a definition of local causality, we can derive Bell inequalities for observable statistics associated with two spacelike separated regions $\mathcal{R}_A$ and $\mathcal{R}_B$ (see figure 1), and that quantum theory does not satisfy the same inequalities. Bell's derivation uses the language of 'beables', but the mathematics requires only that random variables are associated with regions of spacetime, prompting us here to introduce *random fields* to remove the niceties of Bell's concept of 'beables'. Bell introduces a distinction between fields that are 'really supposed to *be* there', the 'beable' fields, and fields that are not real, which is an ontological distinction that is downplayed here (see appendix A for a brief account of 'beables', including Bell's examples of fields that are and are not 'beables').

Almost all experiments that investigate physics at a small scale require the collection of statistics; it is barely possible otherwise to identify any regularities. The mathematical tool that we use as an idealization of statistics is the random variable. At its most general, any indexed set of random variables is a random field, but in the context of physical descriptions that are placed in spacetime, the simplest index set is a lattice of points (see, for example, [2]). This paper will focus, however, on *continuous* random fields (see appendix B and, for example, [3]), for which the index set is the Schwartz space of functions on Minkowski space. A continuous random field can be understood as a random variable-valued distribution. We will focus on continuous random fields because they are very closely parallel to the operator-valued distributions of quantum field theory (see, for example, [4, chapter 2]).

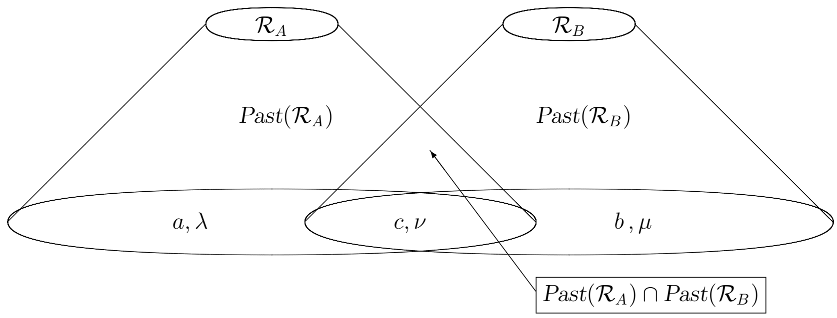

Figure 1. The association of random variables to spacetime regions.

Against Bell [1, chapter 7], Shimony, Horne and Clauser [5, originally 1976] show that if random variables associated with $\mathrm{Past}(\mathcal{R}_A) - \mathrm{Past}(\mathcal{R}_B)$ and with $\mathrm{Past}(\mathcal{R}_B) - \mathrm{Past}(\mathcal{R}_A)$ are correlated with random variables associated with $\mathrm{Past}(\mathcal{R}_A) \cap \mathrm{Past}(\mathcal{R}_B)$, then a model need not satisfy the Bell inequalities. Bell [1, chapter 12, originally 1977] admits this, but finds that random variables associated with $\mathrm{Past}(\mathcal{R}_A) \cap \mathrm{Past}(\mathcal{R}_B)$ have to be correlated with instrument settings in $\mathcal{R}_A$ and in $\mathcal{R}_B$. Arguing that such a requirement is unreasonable, Bell calls it a 'conspiracy' [1, chapter 12, p 103]. Bell's argument and Shimony, Horne and Clauser's comments are brought together in a review article by d'Espagnat [6].

The violation of Bell inequalities by experiment has imposed a moratorium on the construction of classical models, because it is generally acknowledged that the assumptions required to derive Bell inequalities are satisfied for classical two point particle models (and the same is urged here). However, although Bell's argument is quite clearcut for classical particles and for any classical systems that are separated from measurement devices in a well-defined way, the assumptions required to derive Bell inequalities are not usually satisfied for random field models if there are *any* thermal or quantum fluctuations[1]. The correlation that is called 'conspiracy', more than just being natural, is always present for random fields if there are any thermal or quantum fluctuations[2], even though it is not at all natural for classical two-point particle models. Even if this were all, we would not be able to derive Bell inequalities for random fields, but section 3 shows that there are numerous other correlations, *all* of which must also be assumed to be *identically* zero, whereas all of them are nontrivial for random fields if there are any thermal or quantum fluctuations.

The literature on Bell inequalities for 'beables' is quite sparse, and has not changed the general perception that Bell [1, chapter 12] more or less closes the discussion. The more general literature on Bell inequalities, for which the assumptions required to derive Bell inequalities are discussed quite clearly by Valdenebro [9], has come to the same conclusion. Appendix C discusses the relationship between Bell inequalities for 'beables' and the more

[1] A clear distinction between quantum and thermal fluctuations, at least for free fields, is made in [7], and is briefly described in appendix B.

[2] For random fields at thermal equilibrium, two-point correlations at spacelike separation are generally non-zero. At thermal equilibrium, correlations decay more or less exponentially with increasing spacelike separation; for the trivial Gaussian model in three dimensions, for example, the two-point connected correlation function is proportional to e^{-mx}/x as a function of spacelike separation x [8, section 8.1]. These thermal equilibrium correlations at spacelike separation already do not satisfy the assumptions required to derive Bell inequalities. We can also construct non-equilibrium states of the Gaussian model in three dimensions in which correlations decay more or less like the thermal equilibrium state outside a bounded spacetime region $\mathcal{E}$ that contains an experiment, but with arbitrary correlations within $\mathcal{E}$, just because classically we have free control of initial conditions, still more violating the assumptions.

general literature, in the light of section 3. In a later paper, Bell [1, chapter 16, originally 1981] claims to address the question of fields,

> Finally you might suspect that the very notion of a particle, and particle orbit, . . . , has somehow led us astray. Indeed did not Einstein think that fields rather than particles are at the bottom of everything? So the following argument will not mention particles, nor indeed fields, nor any other particular picture of what goes on at the microscopic level,

but the argument he then makes is almost exactly the same argument as is made in the papers referred to above and described in detail below. It is perhaps just because of the abstraction of his argument, with all mention of particles or fields removed, that he does not identify the very different nature of the necessary assumptions that numerous correlations must be precisely zero for the substantially different cases of two point particles, of C^∞ fields, and of random fields.

The fundamental definition required for the derivation of Bell inequalities when random variables associated with spacetime regions are the focus of our attention is of *local causality*. The two competing definitions given by Bell and by Shimony, Horne and Clauser are described in section 2. Section 3 will reproduce Bell's mathematical argument in the form given by d'Espagnat [6] to allow the assumptions required for the derivation to be highlighted. Sections 4 and 5 discuss in detail the various correlations that have to be assumed to be precisely zero; then section 6 shows that the violation of Bell inequalities alone does not justify preferring a quantum field model over a random field model by considering the similarities between a quantum field theoretic Wigner quasi-probability description and a random field probability description of a complete experimental apparatus that violates a Bell inequality. A quantum field model for a complete experimental apparatus requires as much 'conspiracy' as a random field model.

The distinction between classical point particles and random fields lies just in the existence of correlations between random variables associated with $\mathrm{Past}(\mathcal{R}_A) \cap \mathrm{Past}(\mathcal{R}_B)$, $\mathrm{Past}(\mathcal{R}_A) - \mathrm{Past}(\mathcal{R}_B)$ and $\mathrm{Past}(\mathcal{R}_B) - \mathrm{Past}(\mathcal{R}_A)$, which is not usual for two classical point particles that emerge from a central source that lies completely in $\mathrm{Past}(\mathcal{R}_A) \cap \mathrm{Past}(\mathcal{R}_B)$, but is general for random fields. The distinction does *not* lie in finite and infinite degrees of freedom—the assumptions required to derive Bell inequalities are equally not satisfied for a sufficiently fine lattice model in which there are thermal or quantum fluctuations. Because there generally are correlations between random variables associated with regions at spacelike separation in random field models, we may introduce random field models for complete experimental apparatuses where classical two point particle models for a measured system are not adequate and where previously only quantum mechanical models have been thought adequate.

2. Definitions of local causality

First, the random variables (a, λ), (b, μ) and (c, ν) are considered in more detail. They are associated with the disjoint regions $\mathrm{Past}(\mathcal{R}_A) - \mathrm{Past}(\mathcal{R}_B)$, $\mathrm{Past}(\mathcal{R}_B) - \mathrm{Past}(\mathcal{R}_A)$ and $\mathrm{Past}(\mathcal{R}_A) \cap \mathrm{Past}(\mathcal{R}_B)$, respectively (see figure 1). a, b and c are 'non-hidden' [1, chapter 12] random variables, instrument settings that are observed and possibly controlled by the experimenter, while λ, μ and ν are 'hidden' random variables, neither observed nor controlled by the experimenter. As far as classical physics is concerned, the separation of random variables into (a, λ), (b, μ) and (c, ν) is arbitrary, because anything that is hidden today may be revealed tomorrow and whether we observe or record random variables makes no difference, so any derivation of Bell inequalities must be robust under different choices of the separation. There is nothing about the mathematics of section 3 that will determine a separation of random variables into (a, λ), (b, μ) and (c, ν). The only difference between non-hidden random variables and hidden random variables will be that we will integrate over all values of hidden random variables and never integrate over values of non-hidden random variables. It will be useful to consider three choices in this paper: (1) all of a, b, c, λ, μ and ν are non-null sets of random variables; (2) ν is a complete set of random variables, so that c is null; and (3) c is a complete set of random variables, so that ν is null.

The fundamental definition in Bell's derivation of inequalities is that for a *locally causal theory*, for X any random variable associated with a spacetime region $\mathcal{R}_X$, $X_\cap$ *all* of the random variables associated with $\mathrm{Past}(\mathcal{R}_X) \cap \mathrm{Past}(\mathcal{R}_Y)$, X_p *some* of the random variables associated with $\mathrm{Past}(\mathcal{R}_X) - \mathrm{Past}(\mathcal{R}_Y)$ and Y any random variable associated with a spacetime region $\mathcal{R}_Y$ that is spacelike separated from $\mathcal{R}_X$, the conditional probability of X given $X_\cap$ and X_p is statistically independent of Y,

$$p(X|X_\cap, X_p, Y) = p(X|X_\cap, X_p). \tag{1}$$

(In an abuse of notation, we will write X for an event involving the random variable X, where we discuss several events involving the same random variable; they will be denoted by X, X', etc, which may be thought of as shorthand for events E_X, E'_X, etc. This lets us keep close to the notation of the original papers [1, 5, 6].) This definition of local causality is applied a number of times in Bell's derivation of inequalities.

Shimony, Horne and Clauser [5], in contrast, weaken the definition of a locally causal theory, so that for X and Y as above, but for X_P *all* of the random variables associated with $\mathrm{Past}(\mathcal{R}_X)$, the conditional probability of X given X_P is statistically independent of Y,

$$p(X|X_P, Y) = p(X|X_P). \tag{2}$$

The two definitions of a locally causal theory are the same if X_p happens to be *all* the random variables in $\mathrm{Past}(\mathcal{R}_X) - \mathrm{Past}(\mathcal{R}_Y)$, but note that the definition of X_p is so loose that X_p can even be any single random variable associated with $\mathrm{Past}(\mathcal{R}_X) - \mathrm{Past}(\mathcal{R}_Y)$. Equation (1) is presumably supposed to be satisfied for an *arbitrary* choice of random variables as X_p, so it is $X_\cap$ that is characteristic of equation (1). Equation (1) combines equation (2), which is a much more natural definition of local causality for a classical field theory, with a principle that correlations arise only because of common causes. Equation (1) generalizes an idea that two point particles leave a central source at the same time as a common cause of two events in regions $\mathcal{R}_A$ and $\mathcal{R}_B$, entirely reasonable for a classical two-point particle model, to a much more tendentious idea that there must be a common cause of the two events even in a random field model for an experiment.

Equation (2), however, is not strong enough to allow Bell inequalities to be derived. Some of the applications of equation (1) can be replaced by applications of equation (2), but some cannot. As well as the well-known 'no-conspiracy' assumption (discussed in section 4), which prohibits correlations between instrument settings and hidden random variables and is needed whether we adopt equation (1) or equation (2) as our definition of a locally causal theory, section 3 further identifies a 'no-correlation' assumption (discussed in section 5), which prohibits correlations between hidden random variables. The 'no-correlation' assumption is only needed if we adopt equation (2) as our definition of a locally causal theory.

Although equation (2) is a natural definition of local causality, it is not satisfied by the signal local and Lorentz invariant *but analytically nonlocal* dynamics discussed in [7]. With such a dynamics, $p(X|X_P, Y) \neq p(X|X_P)$, even though the nonlocal effects of such a dynamics are restricted to heat-equation-like exponentially reducing tails and signal locality is satisfied.

The derivation of Bell inequalities for random fields also requires an assumption that apparatus must be independent, which is innocuous if almost all random variables are hidden, but is not innocuous if almost all random variables are nonhidden, when significant correlations should be expected. Section 3 identifies an 'independent-apparatus' assumption (discussed in section 4), and shows the 'independent-apparatus' assumption to be closely related to the 'no-conspiracy' assumption. These three assumptions cannot be considered independently. Indeed, which assumptions should be considered to be unsatisfied in a model for an experiment that violates Bell inequalities will depend on what separation there is of random variables into (a, λ), (b, μ) and (c, ν), since the assumptions refer to correlations between non-hidden and non-hidden random variables ('independent-apparatus'), between non-hidden and hidden random variables ('no-conspiracy') and between hidden and hidden random variables ('no-correlation').

3. The derivation of Bell inequalities for random fields

Assumptions that are required to derive Bell inequalities, and that will be discussed in sections 4 and 5, will be indicated by [[*Notes in brackets*]]. Suppose that A and B are random variables associated with regions $\mathcal{R}_A$ and $\mathcal{R}_B$. Recall that the conditional probability $p(X|Y)$ is defined as $p(X|Y) = \frac{p(X,Y)}{p(Y)}$, so that $p(X, Y|Z) = p(X|Y, Z)p(Y|Z)$. Applying this first to $p(A, B, \lambda, \mu, \nu|a, b, c)$, we obtain

$$p(A, B, \lambda, \mu, \nu|a, b, c) = p(A, B|\lambda, \mu, \nu, a, b, c)p(\lambda, \mu, \nu|a, b, c), \tag{3}$$

and applying it again to $p(A, B|\lambda, \mu, \nu, a, b, c)$, we obtain

$$p(A, B|\lambda, \mu, \nu, a, b, c) = p(A|B, \lambda, \mu, \nu, a, b, c)p(B|\lambda, \mu, \nu, a, b, c). \tag{4}$$

Applying equation (1) *or* equation (2), the conditional probability density $p(A|B, \lambda, \mu, \nu, a, b, c)$ is statistically independent of b, μ and B in a locally causal theory, and similarly for the conditional probability density $p(B|\lambda, \mu, \nu, a, b, c)$,

$$p(A|B, \lambda, \mu, \nu, a, b, c) = p(A|a, c, \lambda, \nu), \tag{5}$$

$$p(B|\lambda, \mu, \nu, a, b, c) = p(B|b, c, \mu, \nu), \tag{6}$$

so that

$$p(A, B|\lambda, \mu, \nu, a, b, c) = p(A|a, c, \lambda, \nu)p(B|b, c, \mu, \nu). \tag{7}$$

Using these, the mean of the product AB, given an event (a, b, c), is

$$\begin{aligned} M(a, b, c) &= \iiint \sum_{AB} ABp(A, B, \lambda, \mu, \nu|a, b, c)\, \mathrm{d}\lambda\, \mathrm{d}\mu\, \mathrm{d}\nu \\ &= \iiint \sum_{AB} ABp(A, B|\lambda, \mu, \nu, a, b, c)p(\lambda, \mu, \nu|a, b, c)\, \mathrm{d}\lambda\, \mathrm{d}\mu\, \mathrm{d}\nu \\ &= \iiint \sum_{AB} ABp(A|a, c, \lambda, \nu)p(B|b, c, \mu, \nu)p(\lambda, \mu, \nu|a, b, c)\, \mathrm{d}\lambda\, \mathrm{d}\mu\, \mathrm{d}\nu. \end{aligned} \tag{8}$$

The conditional probability density $p(\lambda, \mu, \nu|a, b, c)$ can also be rewritten by again applying the definition of conditional probability, as

$$\begin{aligned} p(\lambda, \mu, \nu|a, b, c) &= p(\lambda, \mu|\nu, a, b, c)p(\nu|a, b, c) \\ &= p(\lambda|\mu, \nu, a, b, c)p(\mu|\nu, a, b, c)p(\nu|a, b, c). \end{aligned} \tag{9}$$

Applying equation (1), or, through a putative argument provided by Shimony, Horne and Clauser [5] and discussed in section 5, applying equation (2), we can derive

$$p(\lambda|\mu, \nu, a, b, c) = p(\lambda|\nu, a, b, c), \tag{10}$$

$$p(\lambda|\nu, a, b, c) = p(\lambda|\nu, a, c), \tag{11}$$

$$p(\mu|\nu, a, b, c) = p(\mu|\nu, b, c) \tag{12}$$

[[$p(\lambda|\mu, \nu, a, b, c) = p(\lambda|\nu, a, b, c)$ *is the 'no-correlation' assumption; equations* (11) *and* (12) *are further assumptions, which might be called 'no-nonlocal-conspiracy' assumptions, but they will not be directly addressed here*]], so the mean of the product AB, given the event (a, b, c), is

$$M(a, b, c) = \int \overline{A(a, c, \nu)}\ \overline{B(b, c, \nu)}\, p(\nu|a, b, c)\, \mathrm{d}\nu, \tag{13}$$

where $\overline{A(a, c, \nu)}$ is the mean of A averaged over the hidden random variables λ, given the event (a, c, ν), and similarly for $\overline{B(b, c, \nu)}$.

Suppose that A and B satisfy $|A| \leqslant 1$ and $|B| \leqslant 1$, so that $|\overline{A(a, c, \nu)}| \leqslant 1$ and $|\overline{B(b, c, \nu)}| \leqslant 1$. If we also suppose that

$$p(\nu|a, b, c) = p(\nu|c) \tag{14}$$

[[$p(\nu|a, b, c) = p(\nu|c)$ *is the already known 'no-conspiracy' assumption*]], then we can derive, for distinct events a, a', and b, b' for the non-hidden random variables a and b,

$$\begin{aligned} |M(a, b, c) \mp M(a, b', c)| &= \left|\int \overline{A(a, c, \nu)}[\overline{B(b, c, \nu)} \mp \overline{B(b', c, \nu)}]\, p(\nu|c)\, \mathrm{d}\nu\right| \\ &\leqslant \left|\int [\overline{B(b, c, \nu)} \mp \overline{B(b', c, \nu)}]\, p(\nu|c)\, \mathrm{d}\nu\right|, \end{aligned} \tag{15}$$

$$\begin{aligned} |M(a', b, c) \pm M(a', b', c)| &= \left|\int \overline{A(a', c, \nu)}[\overline{B(b, c, \nu)} \pm \overline{B(b', c, \nu)}]\, p(\nu|c)\, \mathrm{d}\nu\right| \\ &\leqslant \left|\int [\overline{B(b, c, \nu)} \pm \overline{B(b', c, \nu)}]\, p(\nu|c)\, \mathrm{d}\nu\right| \end{aligned} \tag{16}$$

[[*Being able to change $a \to a'$ without changing c or b and $b \to b'$ without changing c or a is the 'independent-apparatus' assumption, which is insignificant if a, b and c are just a few inaccurately measured variables, but becomes significant if a, b and c are extensive or complete information about the apparatus.*]] so that

$$|M(a, b, c) \mp M(a, b', c)| + |M(a', b, c) \pm M(a', b', c)| \leqslant 2, \tag{17}$$

because $|\int \overline{B(b, c, \nu)}\, p(\nu|c)\, \mathrm{d}\nu| \leqslant 1$ and $|\alpha| \leqslant 1 \wedge |\beta| \leqslant 1 \Rightarrow |\alpha + \beta| + |\alpha - \beta| \leqslant 2$. In contrast, for two spin-half particles, we can derive the inequalities

$$|M(a, b, c) \mp M(a, b', c)| + |M(a', b, c) \pm M(a', b', c)| \leqslant 2\sqrt{2}, \tag{18}$$

which is essentially the Cirel'son bound [10], but, for a random field model,

if $p(\lambda|\mu, \nu, a, b, c) \neq p(\lambda|\nu, a, b, c)$,	correlation
or $p(\lambda|\nu, a, b, c) \neq p(\lambda|\nu, a, c)$,	nonlocal-conspiracy
or $p(\mu|\nu, a, b, c) \neq p(\mu|\nu, b, c)$,	nonlocal-conspiracy
or $p(\nu|a, b, c) \neq p(\nu|c)$,	conspiracy

then $M(a, b, c)$, $M(a, b', c)$, $M(a', b, c)$ and $M(a', b', c)$ are independent, satisfying only the trivial inequalities

$$|M(a,b,c) \mp M(a,b',c)| + |M(a',b,c) \pm M(a',b',c)| \leqslant 4. \tag{19}$$

Additionally, if we cannot change a, b and c independently, we cannot measure $M(a, b, c)$ and $M(a, b', c)$, for example, because we cannot keep a and c *perfectly* unchanged, making both equations (17) and (19) experimentally unrealizable. All we could measure would be $M(a, b, c)$ and $M(\mathring{a}, b', \mathring{c})$, where $\mathring{a}$ and $\mathring{c}$ are perhaps very close to a and c but not identical.

It is well known that classical local physics allows the maximum value of 4 to be saturated [13–15], but the violation of Bell inequalities has been considered to require unnatural correlations; this paper argues that the required correlations are natural for a classical random field. Classically, quantum mechanics is half way between the conditions for deriving Bell inequalities and the maximum violation, when equation (19) is satisfied as an equality. There must, therefore, be principled constraints on initial conditions in a random field model to ensure that the maximum violation is never observed, as well as to allow some violation. It is an open problem to find a plausible axiom that restricts random field models to the same $2\sqrt{2}$ limit as quantum theory, but there are models in the literature, such as those of Adler [11] and 't Hooft [12], that are approximately within the general structure of a random field but have generally been ignored summarily because of an overzealous belief that violation of the Bell inequalities rules out all classical models.

4. The no-conspiracy and independent-apparatus assumptions

The prohibition of correlations of a with c, and of b with c, the 'independent-apparatus' assumption, is closely related to the 'no-conspiracy' assumption. If we suppose that ν is a complete set of random variables, so that c is null, we can derive in place of equation (13), supposing that equations (10), (11) and (12) are satisfied,

$$M(a,b) = \int \overline{A(a,\nu)}\ \overline{B(b,\nu)}\, p(\nu|a,b)\,\mathrm{d}\nu, \tag{13^ν}$$

which requires that

$$p(\nu|a,b) = p(\nu) \tag{20}$$

for us to be able to derive Bell inequalities. If we take a and b to be only instrument settings at the time of the measurement, with c null, so that ν is *complete* information about the *whole* of $\mathrm{Past}(\mathcal{R}_A) \cap \mathrm{Past}(\mathcal{R}_B)$, the 'no-conspiracy' assumption asserts that instrument settings at the time of the measurement must be completely uncorrelated with the experimental apparatus (which is, after all, almost entirely in $\mathrm{Past}(\mathcal{R}_A) \cap \mathrm{Past}(\mathcal{R}_B)$). Ensuring that instrument settings are *completely* uncorrelated with the experimental apparatus would seem to be a remarkable achievement in a random field theory setting.

Bell argues [1, chapter 12] that the dynamics of a mechanism to choose the instrument settings can be made chaotic enough that, even if there are correlations between (c, ν) and (a, b), the instrument settings may nonetheless be taken to be 'at least effectively free for the purposes at hand'. From a classical point of view, this is a remarkable claim. Either there are correlations in a model for an experiment or there are not. Correlations that are easy to measure at one time are generally not as easy to measure at other times, but the practicality of measuring correlations has no bearing on whether there are correlations, which is in principle unaffected by whether the evolution is chaotic or not.

In any case, a and b being 'free for the purposes at hand' does *not* imply $p(\nu|a, b) = p(\nu)$. A correlation $p(\nu|a, b) \neq p(\nu)$ does not 'determine' a and b (or ν), but only describes a statistical relationship between a, b and ν. If there *is* a correlation between a, b and ν, and we arrange or observe particular statistics for a and b, it just must have been the case that ν had statistics compatible with the correlation, even though we did not control or measure ν. However, ν is not measured—by definition, since it is 'hidden'—so we can only surmise whether there is in fact such a correlation and whether the statistics of ν are compatible with the correlation. How measurements are 'determined' or 'chosen' is independent of $p(\nu|a, b)$, because it is only a record of how the unmeasured variables ν are correlated with the measurement settings; ν might entirely 'determine' or 'choose' the measurement settings a and b or not 'determine' them at all, but have the same correlations with a and b in both cases. As Jaynes puts it, rather forcefully, 'Bell took it for granted that a conditional probability $P(X|Y)$ expresses a physical causal influence, exerted by Y on X' [16].

Bell also argues [1, chapter 12] that 'the disagreement between locality and quantum mechanics is large—up to a factor of $\sqrt{2}$ in a certain sense', and that although the assumptions identified here are not analytically satisfied, nonetheless they are 'nearly' satisfied. First of all, Bell's argument is slightly weakened by the classical limit being either 2 or 4 (Bell omits to mention the latter), depending on whether we accept *all* the standard assumptions, with $2\sqrt{2}$ as the intermediate quantum mechanical limit. More critically, the standard assumptions discussed here are given as analytic equalities, which are unable to elaborate Bell's 'certain sense'. A random field model is so general that it is unclear how the no-correlation, no-nonlocal-conspiracy, no-conspiracy and independent-apparatus assumptions could instead be given as physically justifiable limits on inequality (note that the standard assumptions are problematic just as analytic equalities between probability distributions, since such a relationship cannot be supported by experimental statistics, nor, it seems, by analytic argument).

For a random field model to be empirically adequate, there is no requirement that the assumptions be violated by much, only that the totality of correlations be such that the dynamical evolution will result in the violation of Bell inequalities at the time of measurement. A correlation might be easily measurable at the time of a measurement, but just because the same correlation is almost always not measurable in practice at earlier times does not mean it is zero at earlier times. The chaotic behaviour that Bell invokes to assert that instrument settings cannot be significantly correlated in fact operates rather against Bell's overall argument, since then manifest measured correlations between non-hidden random variables at the time of measurement are all the more likely to correspond to unmeasurable correlations between hidden random variables before the time of measurement.

Making slightly different assumptions, suppose that instead of taking ν to be complete information, we take c to be complete information, so that ν is null. Then we can derive, in place of equation (13), again supposing that equations (10), (11) and (12) are satisfied,

$$M(a, b, c) = \overline{A(a, c)}\ \overline{B(b, c)}. \qquad (13^c)$$

Now to derive equation (17), we have to make only the 'independent-apparatus' assumption, so that we can change $a \to a'$ without changing c (or b) and we can change $b \to b'$ without changing c (or a), with the 'no-conspiracy' assumption playing no role.

In quantum field theory, the Reeh–Schlieder theorem [4] is typically thought very awkward, yet the apparatus dependence it implies is not taken to rule out quantum field theory. Recall that as a consequence of the Reeh–Schlieder theorem we cannot change a quantum field state so that the expected value of a quantum field observable associated with $\mathrm{Past}(\mathcal{R}_A) - \mathrm{Past}(\mathcal{R}_B)$ changes without changing the expected value of almost all quantum field observables associated with both $\mathrm{Past}(\mathcal{R}_A) \cap \mathrm{Past}(\mathcal{R}_B)$ and $\mathrm{Past}(\mathcal{R}_B) - \mathrm{Past}(\mathcal{R}_A)$. Applied in the context of Bell inequalities for random fields, this is just to say that it is impossible in quantum field theory to change $a \to a'$ without changing c at least some of the time, if c is the *complete* set of observables in $\mathrm{Past}(\mathcal{R}_A) \cap \mathrm{Past}(\mathcal{R}_B)$. If instead ν is the complete set of random variables in $\mathrm{Past}(\mathcal{R}_A) \cap \mathrm{Past}(\mathcal{R}_B)$, the Reeh–Schlieder theorem would then be just to say that the 'no-conspiracy' assumption cannot be satisfied in quantum field theory—there must be correlations between ν and (a, b).

If c is not complete information, the correlations of c with a and b should be expected to lessen as c includes fewer and fewer random variables; the correlations should not be expected to become identically zero as soon as c is not the complete set of random variables in $\mathrm{Past}(\mathcal{R}_A) \cap \mathrm{Past}(\mathcal{R}_B)$. Whether we measure or do not measure random variables in the past should not make any difference, in a classical model, to whether violation of Bell inequalities can be observed, but will change the description we give of the correlations we take to cause the violation.

It is unreasonable to expect the 'independent-apparatus' and 'no-conspiracy' assumptions to be satisfied by a random field model when we do not expect them of quantum field theory—to do so is to construct a straw man of a theory. If we insist on a parallel of the Reeh–Schlieder theorem for random fields, we cannot derive Bell inequalities for random fields.

5. The no-correlation assumption

Recall that the 'no-correlation' assumption, equation (10), requires that there are no correlations between the hidden random variables λ and the hidden random variables μ (that are not screened off by ν, a, b and c). There is no empirical way to justify this assumption, simply because it is a condition imposed on random variables that are *by definition* not measured. The preference against correlations between instrument settings and hidden random variables is only tendentiously extensible to justify a prohibition against correlations between hidden random variables.

Shimony, Horne and Clauser [5] argue that

> even though the space-time region in which λ is located extends to negative infinity in time, ν, a, c are *all* the beables other than λ itself in the backward light cone of this region, and μ and b *do* refer to beables with space-like separation from the λ region (their emphasis)

to justify deriving equations (10), (11) and (12) from equation (2) (no additional argument is needed if we take equation (1) as our definition of local causality). This argument relies on the unbounded extent of $\mathrm{Past}(\mathcal{R}_A) - \mathrm{Past}(\mathcal{R}_B)$, so that on a simple interpretation the only random variables associated with the past light cone of $\mathrm{Past}(\mathcal{R}_A) - \mathrm{Past}(\mathcal{R}_B)$, the region $\mathrm{Past}(\mathrm{Past}(\mathcal{R}_A) - \mathrm{Past}(\mathcal{R}_B))$, are c and ν, since a and λ are associated with the region $\mathrm{Past}(\mathcal{R}_A) - \mathrm{Past}(\mathcal{R}_B)$ itself. Consider, however, that for any time slice $\mathrm{Past}(\mathcal{R}_A)_T$ of $\mathrm{Past}(\mathcal{R}_A)$ at time T, we would expect the complete set of random variables associated with $\mathrm{Past}(\mathcal{R}_A)_T$ to determine random variables associated with $\mathcal{R}_A$ (at least probabilistically), but we would *not* expect the complete set of random variables associated with $\mathrm{Past}(\mathcal{R}_A)_T \cap \mathrm{Past}(\mathcal{R}_B)_T$ to determine random variables associated with $\mathcal{R}_A$. As we consider earlier and earlier time slices, the contribution from $\mathrm{Past}(\mathcal{R}_A)_T - \mathrm{Past}(\mathcal{R}_B)_T$ becomes less and less, but the contribution only becomes exactly zero in the infinite past if there is *no* incoming light-like contribution.

Assuming that random variables that determine observables in $\mathcal{R}_A$ and $\mathcal{R}_B$ must be associated with the whole of $\mathrm{Past}(\mathcal{R}_A)$ and $\mathrm{Past}(\mathcal{R}_B)$ goes against the usual structure of classical physics, which almost always takes initial conditions to be associated with a time slice of the past (usually a hypersurface, but at most a spacetime region of finite duration), not to be associated with the whole of the past. The competing definitions of local causality, and the whole derivation of Bell inequalities for random fields, may be put in terms of an arbitrary time slice of Past($\mathcal{R}_A$) and Past($\mathcal{R}_B$). If we associate the random variables λ, μ, ν, a, b and c with a time slice of the backward light cones, not with the whole backward light cones, Shimony, Horne and Clauser's argument fails to justify deriving equations (10), (11) and (12) from equation (2).

Shimony, Horne and Clauser's argument effectively reintroduces common causation, by requiring that there is no causation associated with the region outside Past($\mathcal{R}_A$) $\cap$ Past($\mathcal{R}_B$). As for the no-conspiracy and independent-apparatus assumptions, there is no requirement that the no-correlation assumption be violated by much, only that the totality of correlations of all three kinds be such that the dynamical evolution will result in the violation of Bell inequalities at the time of measurement.

Shimony, Horne and Clauser's argument is contrary to the *intention* behind equation (2), which is that the *complete* causal past of a region determines its present (again, at least probabilistically, even if not deterministically). If we take the trouble to distinguish between equation (1) and equation (2) as definitions of local causality, we cannot argue for equations (10), (11) and (12) in a way that quietly negates the distinction.

6. A quantum field theory approach

We have become used to describing the outcome of Bell violating experiments using a state in a complex four-dimensional Hilbert space, in which many detailed degrees of freedom are integrated out. If we agree, however, that non-relativistic quantum mechanics is a reduction of quantum field theory—as we almost always do—such a state is a reduction of a quantum field state in an infinite-dimensional Hilbert space, which gives the values of quantum field observables associated with the regions $\mathcal{R}_A$ and $\mathcal{R}_B$. If Bell inequalities are violated by observables of a quantum field state, we would certainly attribute the violation to the experimenters' ingenuity in ensuring an appropriate initial quantum field state and making appropriate measurements. For a quantum field state describing an experimental apparatus that violates Bell inequalities, the existence of strong correlations between observables at large spacelike separations is a large part of what singles out such states as special (Bell inequalities are violated slightly even for the vacuum [17], but unmeasurably at large spacelike separations). A quantum field state that describes experimental correlations that measurably violate Bell inequalities at the time of measurement describes correlations in the remote past different from those of the vacuum state, but, as for a random field model, differences from the vacuum state may be difficult to detect in the remote past. In quasi-probability terms, we have to set up a Wigner quasi-distribution over phase space in the past that evolves to a Wigner quasi-distribution over phase space at the time of measurement t_M that violates a Bell inequality in the regions $\mathcal{R}_A$ and $\mathcal{R}_B$.

For an equilibrium state of a random field model, correlations between random variables that violate the assumption of statistical independence at spacelike separation generally decrease more or less exponentially fast with increasing distance, but strong correlations at arbitrarily large distances are possible for non-equilibrium states. Indeed, *absolutely any correlations are allowed in a non-equilibrium initial condition*—initial conditions of low probability of course require greater free energy to set up, but we should not forget how difficult it is to construct an experiment that violates Bell inequalities at large spacelike separations. In a random field model, we have to set up a probability distribution over phase space in the past that evolves to a probability distribution over phase space at time t_M that violates a Bell inequality in the regions $\mathcal{R}_A$ and $\mathcal{R}_B$, but this is no greater 'conspiracy' than is apparent in the full quantum field state for the experiment (to be explicit, note the parallel between the Wigner quasi-distribution description and the classical probability description).

The statistics we observe for random variables in the region $\mathcal{R}_A \cup \mathcal{R}_B$ are no more than classical initial conditions. We cannot rule out any classical dynamics, whether local or nonlocal, without Bell's other assumptions, which amount to a claim that unobserved initial conditions at earlier times cannot, for *a priori* reasons, be correlated in such a way that the observed initial conditions at the time of measurement are as we observe them. There are often significant reasons for preferring a quantum field model over a random field model, such as ease of computation, years of familiarity and the analytical power of the mathematics of Hilbert spaces, but the violation of Bell inequalities is not conclusive.

The correlations we have discussed here commit us to very little, if we take an equally empiricist approach to random fields as we take to quantum fields: correlations just exist; we do not have to assume that they are caused by common (or any other kind of) causes. Classical physics has generally taken initial conditions to be more or less explained by earlier initial conditions, with no final explanation being essential.

7. Discussion

We have described the previously identified difference between Bell's definition of a locally causal theory, which insists that correlations have to be the result of common causes, and Shimony, Horne and Clauser's definition, which does not. The assumption that there *is* a common cause for separated events is an *a priori* constraint on unmeasured initial conditions at earlier times. This is quite a natural assumption for a classical two-particle model, because the two particles are imagined to be emitted from a single point in the past, but it is a strong and unjustifiable assumption for a random field model. We have described numerous assumptions that random variables may not be correlated with other random variables, all of which are necessary for Bell inequalities to be derived, but none of which are generally satisfied for a random field in the presence of thermal or quantum fluctuations.

We have also seen that the opprobrium of 'conspiracy' as much applies to quantum fields as it does to random fields. We could argue from this that quantum field models should as much be rejected as random field models, but it seems more appropriate for physics to admit both. The long-standing moratorium on construction of classical models loses most of its justification if we allow ourselves to use the resources of random fields.

To temper the localism of this paper, repeating the caution given in section 2, a random field model that reproduces the phenomenological success of a quantum field model has to have the same propagator as the given quantum field model, which in classical terms is nonlocal even while preserving signal locality and being Lorentz invariant [7]. Thus the very first assumption that is needed to derive Bell inequalities for a random field model, of locality in a classical dynamical sense, is not satisfied in any realistic model, even though an assumption of signal locality is satisfied. It is well known that quantum field theory is nonlocal in the sense of Hegerfeldt [18], while nonetheless preserving signal locality [19]. The violation of Bell inequalities can be modelled by entirely local random fields, but leaves an awkward question of how the nonlocal correlations might have been established in the first place (that is, how did the 'conspiracy' arise?), which finds a relatively more natural answer if the propagator of a random field model is nonlocal.

Although the principal argument of this paper is that correlations in random field models generally do not satisfy the assumptions necessary to derive Bell inequalities, empirically adequate random field models will often have to include a detailed description of the measurement apparatus, which may well not be easy to construct. If thermal properties of a measurement apparatus have to be taken into account explicitly in a quantum field model to ensure empirical adequacy, however, then a random field model should be no more complex than the quantum field model. A random field approach might also connect better with general relativity because of their shared classicality, but in my investigations so far the connection seems to be as awkward as it is for quantum field theory. Quantum theory is of course generally more easily usable than a random field model whenever a finite-dimensional Hilbert space is empirically adequate.

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Appendix A. Beables

The distinction Bell makes between 'beable' fields and 'non-beable' fields, and the ontology that Bell introduces, are not significant for the approach of this paper. The only aspect of Bell's idea of 'beables' that matters, both to his and to my mathematical derivation, is the attachment of random variables to regions of spacetime. Nonetheless, Bell gives the name *The theory of local beables* to the paper that is the principal stem of the literature [1, chapter 7], so there should be a brief discussion here about beables.

As a first example, Bell distinguishes between the electromagnetic fields **E** and **H** as 'physical' and the electromagnetic potentials ϕ and **A** as 'non-physical'; Bell emphasizes that the connection is just a mathematical convenience that is 'not really supposed to *be* there' (from a random field perspective, the reason for discounting the 4-potential A_μ is more mathematical than physical or ontological—a classical connection A_μ cannot be averaged by integration over a region, so it cannot be extended to a distribution that generates smeared observables, in contrast to the electromagnetic field). In the next paragraph Bell also denies that the wavefunction is a 'beable', for a somewhat different reason, the "collapse of the wavefunction' on 'measurement", which he describes as 'one of the apparent nonlocalities of quantum mechanics'. Bell resolves his ontological difficulties by claiming that the 'odd behaviour' of the wavefunction is acceptable if we take the wavefunction also to be only a mathematical convenience. It is not quite clear what we should take the common feature of these examples to be, except perhaps the odd behaviour (the electromagnetic potential is guilty only of 'funny behaviour'), which is the signal for mathematics to be taken to be only a convenience instead of real.

Finally, he gives a description of what is important both for his and my purposes, 'We will be particularly concerned with *local* beables, those which (unlike for example the total energy) can be assigned to some bounded space-time region' (his emphasis). It is manifest from the mathematics of section 3 that all that matters *for the mathematics* is the association of random variables with bounded regions of spacetime.

Appendix B. Continuous random fields

For the purposes of this paper, continuous random fields can most appropriately be understood either as random variable-valued distributions or, in a Koopman–von Neumann type of approach, as a commutative quantum field (but see also [3]). We introduce either a random variable-valued linear map, $\chi : f \mapsto \chi_f$, or an operator-valued linear map $\hat{\chi} : f \mapsto \hat{\chi}_f$, with the trivial commutator $[\hat{\chi}_f, \hat{\chi}_g] = 0$ whatever the spacetime relationship between Schwartz space functions f and g (a Schwartz space function $f(x)$ is infinitely often differentiable and decreases as well as its derivatives faster than any power as x moves to infinity in any direction [4, section II.1.2]). The difference between these two approaches is mostly notational, but operator-valued distributions are used in this appendix to emphasize the similarities to and differences from quantum fields. In contrast to the random field, for a quantized Klein–Gordon field, an operator-valued linear map $\hat{\phi} : f \mapsto \hat{\phi}_f$, the commutator $[\hat{\phi}_f, \hat{\phi}_g] = (g, f) - (f, g)$ is zero when f and g have spacelike separated supports. (g, f) is a manifestly Lorentz invariant Hermitian inner product on the Schwartz space,

$$(g, f) = \hbar \int \frac{\mathrm{d}^4k}{(2\pi)^4} 2\pi\delta(k^\mu k_\mu - m^2)\theta(k_0)\tilde{g}^*(k)\tilde{f}(k) \tag{B.1}$$

$$= \hbar \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{\tilde{g}^*(\mathbf{k})\tilde{f}(\mathbf{k})}{2\sqrt{\mathbf{k}^2 + m^2}}. \tag{B.2}$$

Although we usually define the vacuum state of the quantized Klein–Gordon field in terms of the trivial action of a creation operator, we can equally well define it by the characteristic function

$$\langle 0|\, \mathrm{e}^{\mathrm{i}\lambda\hat{\phi}_f} |0\rangle = \mathrm{e}^{-\frac{1}{2}\lambda^2(f,f)}, \tag{B.3}$$

which is enough to fix the Wightman functions of the quantum field (by linearity, $\mathrm{e}^{\mathrm{i}\lambda\hat{\phi}_f + i\mu\hat{\phi}_g} = \mathrm{e}^{\mathrm{i}\hat{\phi}_{\lambda f+\mu g}}$, which we can use to construct a multivariate characteristic function). Other sectors can be constructed by changing the right-hand side, which, as well as thermal sectors, include 'extra quantum fluctuation' sectors [20],

$$\langle 0|\, \mathrm{e}^{\mathrm{i}\lambda\hat{\phi}_f} |0\rangle = \mathrm{e}^{-\frac{1}{2}\alpha\lambda^2(f,f)}, \qquad \alpha > 1 \tag{B.4}$$

($\alpha < 1$ is not a state over the algebra of observables of the quantized Klein–Gordon field). Analogously, we can define a state of the random field $\hat{\chi}_f$ by the characteristic function

$$\varphi_0(\mathrm{e}^{\mathrm{i}\lambda\hat{\chi}_f}) = \mathrm{e}^{-\frac{1}{2}\lambda^2(f,f)}. \tag{B.5}$$

For this random field state, all joint probability densities over observables $\hat{\chi}_{f_1}, \hat{\chi}_{f_2}, \ldots, \hat{\chi}_{f_n}$ are identical to the equivalent joint probability densities for the quantized Klein–Gordon field, whenever the quantum field observables are also compatible. Self-adjoint functions of non-commuting observables of the quantized Klein–Gordon field such as $\hat{\phi}_f\hat{\phi}_g + \hat{\phi}_g\hat{\phi}_f$ will of course generally have different probability densities from their random field equivalents.

The algebras of observables are not the same, but the state over the classical algebra is sufficiently similar to the vacuum state over the quantum algebra to make it reasonable to call the random field state a presentation of 'quantum fluctuations'. Certainly the amplitude of the fluctuations of the random field is controlled by $\hbar$ and the fluctuations are distinct from the thermal fluctuations of a classical Klein–Gordon field, which can be presented as $\varphi_C(\mathrm{e}^{\mathrm{i}\lambda\hat{\chi}_f}) = \mathrm{e}^{-\frac{1}{2}\lambda^2(f,f)_C}$, with the Lorentz non-invariant inner product

$$(g, f)_C = \mathsf{k}_B T \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{2\pi\delta(k^\mu k_\mu - m^2)\theta(k_0)}{\frac{1}{2}k_0} \tilde{g}^*(k)\tilde{f}(k) \tag{B.6}$$

$$= \mathsf{k}_B T \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{\tilde{g}^*(\mathbf{k})\tilde{f}(\mathbf{k})}{(\mathbf{k}^2 + m^2)}. \tag{B.7}$$

This thermal state can be presented either with a trivial commutator $[\hat{\chi}_f, \hat{\chi}_g] = 0$ or with the commutator $[\hat{\chi}_f, \hat{\chi}_g] = (g, f)_C - (f, g)_C$, depending on whether we wish to use models in which idealized measurements are always compatible or generally not compatible at timelike separation because of *thermal* fluctuations (see [20]).

The difference between quantum fields and random fields can be taken to be only a different attitude to idealized measurements; actual measurements can be described in terms of either. The empirical principle that justifies the implicit description of quantum fluctuations that underlies quantum theory is our apparent inability to reduce the quantum fluctuations of our measurement apparatuses, in contrast to the almost universal minimization of thermal fluctuations in precision experiments. Even if this empirical principle is unbroken, however, we can still model quantum fluctuations and their effects explicitly instead of implicitly, just as we model thermal fluctuations and their effects explicitly when we have to. It is perhaps a conceptual advantage that classical random fields explicitly describe thermal and quantum fluctuations in the same way. A mathematical model is valuable as a mental image of the world, not necessarily as how the world really is; we can *imagine* what the results of measurements might be if we had classically ideal measurement devices, even if we do not have any.

More mathematics and discussion can be found in [7], [20].

Appendix C. Bell's original approach

The more general literature on Bell inequalities more or less follows Bell's original approach [1, chapter 2], in that λ, μ and ν are not distinguished by their spacetime associations, but all hidden random variables are instead written as a single set Λ. Also, a and b are generally taken to be settings at the time of measurement, so that they are associated with $\mathcal{R}_A$ and $\mathcal{R}_B$ instead of with $\mathrm{Past}(\mathcal{R}_A) - \mathrm{Past}(\mathcal{R}_B)$ and $\mathrm{Past}(\mathcal{R}_B) - \mathrm{Past}(\mathcal{R}_A)$. Finally, c is generally taken to be null. However, only the lack of distinction between λ, μ and ν makes a significant difference. Following the analysis of section 3, the more general literature (rationally reconstructed, since many different notations are used) writes

$$M(a,b) = \int \sum_{AB} ABp(A, B, \Lambda|a, b)\, \mathrm{d}\Lambda \tag{C.1}$$

$$= \int \sum_{AB} ABp(A, B|\Lambda, a, b)p(\Lambda|a, b)\, \mathrm{d}\Lambda \tag{C.2}$$

$$= \int \sum_{AB} ABp(A|\Lambda, a, b)p(B|\Lambda, a, b)p(\Lambda|a, b)\, \mathrm{d}\Lambda \tag{C.3}$$

$$= \int \sum_{AB} ABp(A|\Lambda, a)p(B|\Lambda, b)p(\Lambda|a, b)\, \mathrm{d}\Lambda, \tag{C.4}$$

in which the assumptions required to derive equations (C.3) and (C.4) correspond to Jarrett's 'completeness' and 'locality' respectively [21] (or 'outcome independence' and 'parameter independence' in Shimony's terminology [22]). To allow Bell's original approach, it has to be assumed further that $p(\Lambda|a, b) = p(\Lambda)$ ('no-conspiracy'). If we take Λ to be the microstate of a measured system, as we typically do if we think we are measuring the state of two classical point particles, correlation of Λ with a and b represents contextuality of the measured system state, which is generally taken to be anathema. In a random field context, however, it is far more natural to take Λ to be the microstate of the whole experimental apparatus, because for general random fields *there is no natural way to draw an exact boundary* between what would usually be termed the measurement device and the measured system, so that it seems that Λ, when considered in the fullest possible detail, *must* be correlated with a and b (but this does not claim that Λ causes or does not cause a and b).

The derivation of Bell inequalities given in section 3 subsumes the discussion that is possible if we do not distinguish λ, μ and ν, so we will not further pursue the limited approach of this appendix.

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